ACC for LCT IV

Last time:

Thm A_n: ACC for LCT

Thm B_n: Upper bounds on vol.

Thm C_n: Birational Boundedness

Thm D_n: ACC for num. trivial pairs

- $D_{n-1} \Rightarrow A_n$
- $D_{n-1} + A_{n-1} \Rightarrow B_n$
- $C_{n-1} + A_{n-1} + B_n \Rightarrow C_n$

Lemma $D_{n-1} + C_n \Rightarrow D_n$

Recall D_{n-1}: Fix DCC set J

$\exists$ finite set $J_0 \subseteq J$ s.t. if (S, Θ) satisfies:

- S proj of dim $n-1$
- $\mathrm{coeff}(\Theta) \subseteq J$ (will set $J = D(I)$)
- (S, Θ) is lc
- $K_S + \Theta \equiv 0$

Then $\mathrm{coeff}(\Theta) \subseteq \underline{J_0}$.

C_n: $\exists$ constant m s.t. if (Y, Γ) satisfies:

- Y proj of dim n
- $\mathrm{coeff}(\Gamma) \subseteq I$ (DCC set)
- (Y, Γ) is lc
- $K_Y + \Gamma$ is big

Then $\phi_{m(K_Y + \Gamma)}$ is birational.

Pf of Lem 8.1

(X, Δ) satisfies D_n.

come from adjunction

Define $I_1 = \{ c \in I : \frac{m-1+f+kc}{m} \in J_0 \text{ for some } m, k \in \mathbb{N},\ f \in J = D(I) \}$

By Lem 5.2 : I_1 is finite.

For $1 \le l \le m$, let $a_l := \begin{cases} \text{largest element of } [\frac{l}{m}, \frac{l+1}{m}) \cap I, & \text{if any} \\ 1 & \text{else.} \end{cases}$

$I_2 = \{ a_l : 1 \le l \le m \}$. finite.

Claim: $\operatorname{coeff}(\Delta) \subseteq I_1 \cup I_2 \cup \{1\}$.

Pf of claim: Let $\pi : Y \to X$ be a dlt mod s.t.

- Y is $\mathbb{Q}$-fact.
- If $K_Y + \Gamma = \pi^*(K_X + \Delta)$, then

$$\Gamma = \pi_*^{-1} \Delta + \operatorname{Ex}(\pi) \quad \text{and} \quad (Y, \Gamma) \text{ is dlt.}$$

We can replace (X, Δ) by (Y, Γ) to assume (X, Δ) satisfies above.

Let B be any prime comp. of Δ with $\operatorname{coeff}_B(\Delta) = i < 1$ (need to show $i \in I_1 \cup I_2$).

Case 1: B intersects $\lfloor \Delta \rfloor$.

Let S be normalization of this comp. of $\lfloor \Delta \rfloor$. Adjunction says

$$(K_X + \Delta)|_S = K_S + \Theta,$$

(c = coeff of comp. of Θ)

where $\operatorname{coeff}(\Theta) \subset \{ \frac{m-1+f+kc}{m} \} \in D(I)$.

(S, Θ) satisfies $D_{n-1} \Rightarrow \operatorname{coeff}(\Theta) \subseteq J_0$.

$$\Rightarrow \operatorname{coeff}(\Delta) \subset I_1$$

Case 2: B doesn't intersect $\lfloor\Delta\rfloor$.

Run $(\underbrace{K_X+\Delta-iB}_{\text{not pseff}})$-MMP:

$\Rightarrow$ We get
$$\begin{tikzcd} X \arrow[r, dashed, "f"] & Y \arrow[d, "\text{MFS}"] \\ & Z \end{tikzcd}$$

$K_X+\Delta\equiv 0 \Rightarrow$ Every step of MMP is B-positive.

$\Rightarrow \lfloor\Delta\rfloor$ cannot be contracted.

B is not contracted (otherwise $f_*(K_X+\Delta-iB)\equiv 0$)

$\Rightarrow$ We can replace (X,Δ) with $(Y, f_*\Delta)$

Thus, we can assume $X\to Z$ MFS and B dominates Z.

If $Z\neq pt$, then replace (X,Δ) with general fiber and induct.

If $Z=pt$, then $\rho(X)=1$. and B ample, $\lfloor\Delta\rfloor=0$, (X,Δ) klt.

Suppose $j\in I$, $j>i$. Let $\pi: Y\to X$ be a log resol of (X,Δ).

$\Delta_Y = \pi_*^{-1}\Delta$ (strict transform)

$B_Y = \pi_*^{-1}B$

$E = \mathrm{Ex}(\pi)$

Define $\Gamma := \Delta_Y + E + (j-i)B_Y$ $\quad(\mathrm{coeff}_{B_Y}(\Gamma)=j)$.

(Y,Γ) lc, $\mathrm{coeff}(\Gamma)\subseteq I$.

Write

$$K_Y+\Gamma = \pi^*(\underbrace{K_X+\Delta}_{\text{klt}}) + F, \text{ then } F\geq 0 \text{ and } \pi\text{-exceptional}.$$

Pick $\varepsilon > 0$ s.t. $F \geqslant \varepsilon E$.

let $\delta > 0$ s.t. $(j-i)B_Y + \varepsilon E > \delta \pi^* B$.

Then $K_Y + \Gamma = \underbrace{(K_Y + \Delta_Y + (1-\varepsilon)E)}_{\geq \pi^*(K_X + \Delta)} + \underbrace{(j-i)B_Y + \varepsilon E}_{> \delta \pi^* B}$

$\geqslant \pi^*(\underbrace{\underbrace{K_X + \Delta}_{\equiv 0} + \delta B}_{\text{ample}})$. is big.

Thm C_n $\Rightarrow$ $\exists$ const m s.t. $\Phi_{m(K_Y+\Gamma)}$ is bir.

$\Rightarrow K_Y + \frac{1}{m}\lfloor m\Gamma \rfloor$ is big.

$\Rightarrow K_X + \frac{1}{m}\pi_* \lfloor m\Gamma \rfloor$ is big

$\Rightarrow K_X + \frac{1}{m}\lfloor m(\Delta + (j-i)B)\rfloor$ is big.

However, $K_X + \Delta \equiv 0$ is not big

$\Rightarrow \frac{1}{m}\lfloor m(\Delta + (j-i)B)\rfloor > \Delta$.

$\Rightarrow \frac{1}{m}\lfloor mj \rfloor > i$ (compare coeff of B)

If $i \in [\frac{\ell}{m}, \frac{\ell+1}{m})$, then $j \geqslant \frac{\ell+1}{m}$.

$\Rightarrow i = a_\ell \in I_2$. $\blacksquare$

Thm 1.1 Fix $n \in \mathbb{N}_+$, $I \subseteq [0,1]$, $J \subseteq \mathbb{R}_+$, I, J satisfy DCC.

Then $\mathrm{LCT}_n(I,J) = \{ \mathrm{lct}(X, \Delta; M) : \mathrm{coeff}(\Delta) \in I,\ \mathrm{coeff}(M) \subseteq J$

(X,Δ) lc, $\dim X = n \}$

satisfies ACC.

Pf. Suppose $c_1 < c_2 < \cdots \in \mathrm{LCT}_n(I,J)$

(X_k, Δ_k) lc pair and M_k $\mathbb{R}$-Cartier

$\mathrm{coeff}(\Delta_k) \subseteq I$, $\mathrm{coeff}(M_k) \subseteq J$, s.t.

$$c_k = \mathrm{lct}(X_k, \Delta_k; M_k).$$

Let $\Theta_k = \Delta_k + c_k M_k$, then (X_k, Θ_k) is strictly lc.

Let V_k be non-klt center of (X_k, Θ_k) contained in M_k.

Assume every comp. of Θ_k contains V_k.

Thm A $\Longrightarrow$ $\exists$ finite set $K_0 \subseteq \{ i + c_k j : i \in I, k \in \mathbb{N}, j \in J \}$

s.t. $\mathrm{coeff}(\Theta_k) \subseteq K_0$.

By passing to subseq. we can assume

$i_k + c_k j_k$ is constant. a

$$\Longrightarrow c_k = \frac{a - i_k}{j_k}.$$

$\{(a - i_k)\}$ is ACC, $\{\frac{1}{j_k}\}$ ACC $\Rightarrow \{c_k\}$ is ACC, contradiction!

$\blacksquare$

Thm 1.3 Fix $n \in \mathbb{N}_+$, $I \subseteq [0,1]$ DCC.

$$\mathfrak{D} = \{ (X, \Delta) \text{ lc} : \dim X = n, \operatorname{coeff}(\Delta) \subseteq I \}$$

Then $\exists\ \delta > 0$ and $m \in \mathbb{N}_+$ s.t.

(1) $\{ \operatorname{vol}(X, K_X + \Delta) : (X, \Delta) \in \mathfrak{D} \}$ satisfies DCC.

If further $K_X + \Delta$ is big, $(X, \Delta) \in \mathfrak{D}$, then

(2) $\operatorname{vol}(X, K_X + \Delta) \geq \delta$

(3) $\phi_{m(K_X + \Delta)}$ is birational.

Pf: (1) $\Rightarrow$ (2)

Thm C $\Rightarrow$ (3).

Fix $V > 0$. Consider

$$\mathfrak{D}_V = \{ (X, \Delta) \in \mathfrak{D} : 0 < \operatorname{vol}(K_X + \Delta) \leq V \}.$$

(3) + Thm 3.5.2 $\Rightarrow$ $\{ \operatorname{vol}(K_X + \Delta) : (X, \Delta) \in \mathfrak{D}_V \}$ is DCC.,

which implies (1).

$\blacksquare$

Thm 1.6 Fix $n \in \mathbb{N}_+$, $\delta, \varepsilon > 0$. Let $\mathfrak{D}$ be the set of (X, Δ) s.t.:

- X proj of dim n
- $K_X + \Delta$ ample
- $\operatorname{coeff}(\Delta) \geq \delta$
- Total log discrep $(X, \Delta) > \varepsilon$.

If $\mathfrak{D}$ is log birationally bounded, then $\mathfrak{D}$ is bounded.

⌈ Recall log bir bounded:

$\exists\ (Z, B) \longrightarrow T$ projective, T finite type $/\mathbb{C}$, B divisor on Z. $\operatorname{coeff}(B) = 1$

$\forall\ (X, \Delta) \in \mathfrak{D}$, $\exists\ t \in T$ s.t. $(Z_t, B_t) \overset{f}{\dashrightarrow} (X, \Delta)$ and

$$B_t \geq \operatorname{Ex}(f) + f_*^{-1}\Delta.$$

(Boundedness : require $f : (Z_t, B_t) \longrightarrow (X, \Delta)$ is an isom.). ⌋

<u>Pf</u> Step 1 Reduction to SNC pairs.

We may assume $(Z, B) \to T$ is SNC over T (stratify T)

and every stratum of (Z, B) has irred fibers over T;

T is smooth and connected.

Note $\delta \leq 1 - \varepsilon$. Let $\Phi = (1-\varepsilon) B$.

(Z, Φ) is klt $\Rightarrow$ only finitely many val v s.t.

$$A(Z, \Phi; v) \leq 1.$$

$\Rightarrow$ Can find seq. of blow-ups $Y \to Z$

which extracts all div E with $A(Z, \Phi; E) \leq 1$, Let $(X,\Delta) \in \mathfrak{D}$, with $(Z_t, B_t) \overset{f}{\dashrightarrow} (X,\Delta)$ birational.

<u>Claim</u> $Y_t \longrightarrow Z_t \dashrightarrow X$ is a bir. contraction.

Pf of Claim:

$$
\begin{tikzcd}
 & W \arrow[dl, "q"'] \arrow[dr, "p"] & \\
Z_t \arrow[rr, dashed, "f"] & & X
\end{tikzcd}
$$

$E_p := \mathrm{Ex}(p)$ (on W)

$G := p_*^{-1}\Delta + (1-\varepsilon)\cdot E_p$ (on W)

Write $K_W + G = p^*(K_X+\Delta) + F_p$

Total log discrep $(X,\Delta) > \varepsilon \Rightarrow F_p \geqslant 0$ and p-ex.

$\Psi := q_* G = f_*^{-1}\Delta + q_*(1-\varepsilon)E_p$.

(on Z_t)

Then $\Psi \leqslant (1-\varepsilon)B_t$ because
- $q_*(1-\varepsilon)E_p \leqslant (1-\varepsilon)B_t$
- $f_*^{-1}\Delta \leqslant (1-\varepsilon)B_t$ $(\mathrm{coeff}(\Delta) \leq 1-\varepsilon)$

Write $q^*(K_{Z_t} + \overset{q_*G}{\overset{\|}{\Psi}}) := K_W + G + F_q$, F_q is q-ex.

$$\overset{(\ast)}{=} p^*(K_X+\Delta) + F_p + F_q.$$

$p^*(K_X+\Delta)$ is nef, hence q-nef, so $(\ast)$: $-(F_p+F_q)$ is q-nef.

By negativity lemma, $F_p + F_q \geqslant 0$.

Suppose D is any prime div. on X.

$$
\begin{aligned}
A(Z_t, \Phi_t = (1-\varepsilon)B_t; D) &\leqslant A(Z_t, \Psi; D) \\
&\leqslant A(X,\Delta; D) \quad \text{by } (\ast) \\
&\leqslant 1
\end{aligned}
$$

$\Rightarrow$ D lives on Y_t. $\square$

Now replace (Z, B) by $(Y, B_Y + \mathrm{Ex}(Y \to X))$ to assume that

all $Z_t \dashrightarrow X$ are birational contractions.

$$
\begin{tikzcd}
 & W \arrow[dl, "q"'] \arrow[dr, "p"] & \\
Z_t \arrow[rr, dashed, "f"] & & X
\end{tikzcd}
$$

$p_* q^* = f_*$ and $(\circledast)$ shows

$$f_*(K_{Z_t} + \Psi) = K_X + \Delta + p_*(F_p + F_q).$$

$$\shortparallel$$

$K_X + \Delta$ ample

$\Rightarrow$ $F_p + F_q$ is p-ex.

$\Rightarrow$ $f: (Z_t, \Psi) \dashrightarrow \boxed{(X, \Delta) \text{ is the lc model of } (Z_t, \Psi)}$.

Step 2 Deformation invariance for SNC pairs

Note: $K_{Z_t} + \Psi$ is big by $(\circledast)$, $\Psi \le (1-\varepsilon) B_t$

$\mathrm{coeff}(\Psi) \geq \delta$.

Since B has finitely many comp., we may assume

$$\delta B_t \le \Psi \le (1-\varepsilon) B_t$$

⌈ Idea: if Z_t is constant family,

then BCHM $\Rightarrow$ finitely many lc models for (Z_t, Ψ)

where $\Psi \in [\delta B_t, (1-\varepsilon) B_t]$.

Need: lc models for (Z_t, Ψ) form a family.

It's not hard to guess this is ~~the~~ some lc model of total space Z. ⌋

By BCHM Cor 1.1.5, can find fin. many contractions

$f_i : Z \dashrightarrow W_i$ over T s.t.

$\Xi \in [\delta B, (1-\varepsilon) B]$, and $g : Z \dashrightarrow W$ is the lc model for (Z, Ξ)

then $g = f_i$ for some i.

Suffices: if $\Xi|_{Z_t} = \Psi$ and $g : (Z, \Xi) \dashrightarrow W$ is the lc model

then $g|_{Z_t} = f$.

$\iff$ $g|_{Z_t}$ is a lc model for (Z, Ψ).

$\iff$ $R(Z, k(K_Z + \Psi)) = \bigoplus_{m \geq 0} H^0(Z, mk(K_Z + \Psi))$

is finitely generated.

By the proof of HMX 13, Thm 1.8:

$$\underline{R(Z, k(K_Z + \Xi}))\twoheadrightarrow \underline{R(Z_t, k(K_{Z_t} + \Psi))}$$

surjective for k suff. divisible.

$\boxtimes$

<u>Cor 1.7</u> Fix $n \in \mathbb{N}_+$, $\varepsilon, \delta > 0$, $I \subseteq [0,1]$ DCC.

Let $\mathfrak{D}$ be the set of (X, Δ) where

- X proj of dim n
- $\mathrm{coeff}(\Delta) \subseteq I$
- Total log discrep $(X, \Delta) > \varepsilon$
- $K_X + \Delta \equiv 0$
- $-K_X$ is ample

Then $\mathfrak{D}$ is a bounded family.

Pf Thm A $\Rightarrow$ $\mathrm{coeff}(\Delta) \subseteq I_0$ finite set.

Thm B $\Rightarrow$ $\mathrm{vol}(X, \Delta) < C$ constant.

Let D = sum of comp. of Δ. then $\Delta \leq (1-\varepsilon) \cdot D$.

$D \leq r\Delta$, $r = r(I_0)$

$$K_X + D \equiv D - \Delta \geqslant \left(\frac{1}{1-\varepsilon} - 1\right) \cdot \Delta \Rightarrow K_X + D \text{ is big}$$

$$\mathrm{vol}(K_X + D) = \mathrm{vol}(D - \Delta) \leqslant \mathrm{vol}(D) \leqslant r^n \mathrm{vol}(\Delta) < r^n C.$$

$\pi : Y \to X$ be a log resol. of (X, Δ)

$G := \mathrm{supp}\, \pi_*^{-1}\Delta + \mathrm{Ex}(\pi)$. , (Y, G) SNC

Pick $\eta > 0$ s.t. $(X, (1+\eta)\Delta)$ is klt and has log discrep $> \frac{\varepsilon}{2}$.

$K_X + (1+\eta)\Delta$ is ample.

$K_Y + \Gamma = \pi^*(K_X + (1+\eta)\Delta)$ is big

$\Gamma \leq G$ $\big((X, (1+\eta)\Delta)$ klt$\big)$ $\Rightarrow$ $K_Y + G$ is big.

$\mathrm{vol}(K_Y + G) \leq \mathrm{vol}(K_X + D)$ bdded above.

Thm 1.3 $\Rightarrow \exists\, m$ s.t. $\phi_{m(K_Y+\Gamma)}$ is bir. $\Big\}$ $\xrightarrow[\text{Thm 3.1}]{[\text{HMX B}]}$

$\mathrm{vol}(K_Y+G)$ bdded above

$\{(Y,G)\}$ is log bir. bdded.

$\Rightarrow \{(X,\Delta)\}$ is log bir bdded.

To apply Thm 1.6, need to perturb Δ.

For each $(X,\Delta)\in\mathfrak{D}$, pick general ample H (Cartier)

Pick $\delta' \geq \delta$ s.t. $\delta' \notin I_0$. (finite)

Then by same argument, $\{(X,\Delta+\delta' H)\}$ is log bir bdded.

Thm 1.6 $\Longrightarrow \{(X,\Delta+\delta' H)\}$ is bounded by $(Z,B)\to T$.

$\Longrightarrow \{(X,\Delta)\}$ is bounded by $(Z,B')\to T$

where $B' = B -$ (comp. in B with coeff δ').

$\boxtimes$

Cor 1.10

Fix $I \subseteq [0,1]$ ACC, $n \in \mathbb{N}$. Then the Fano spectrum

$$R_n(I) := \{ r \in \mathbb{R} : r \text{ is the Fano index of } (X,\Delta),$$
$$(X,\Delta) \text{ proj lc pair of dim } n, \ -(K_X+\Delta) \text{ ample},$$
$$\text{coeff}(\Delta) \subseteq I \}$$

$\parallel$ $\sup\{ r : rH \sim_{\mathbb{Q}} -(K_X+\Delta)$, H ample Cartier $\}$.

satisfies ACC.

Pf Suppose $r_1 < r_2 < \cdots \in R_n(I)$.

By cone thm, $-(K_X+\Delta)\cdot C < 2n \Rightarrow r_i < 2n$.

Let $(X_i, \Delta_i) \in \mathfrak{D}$ s.t. $\dot{-}(X_i+\Delta_i) \sim r_i H_i$

$(X,\Delta) = (X_i, \Delta_i)$, $H = H_i$, $r = r_i$

By an effective bpf thm (Fujino), $\exists\, m$ s.t. $|mH|$ is bpf.

Pick general $D \in |mH|$. Then

$$K_X + \Delta + \frac{r}{m} D \sim_{\mathbb{R}} 0 \qquad \text{is lc.}$$

$$\text{coeff}(\Delta + \frac{r}{m}D) \subseteq I \cup \left\{ \frac{r_i}{m}, i \in \mathbb{N} \right\} \text{ is DCC}.$$

Thm 1.4 $\Rightarrow$ $\text{coeff}(\Delta + \frac{r}{m} D) \subseteq I_0$ finite.

$\blacksquare$